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RESEARCH ARTICLE

Relativistic Treatment of Quantum Mechanical Gravitational-Harmonic Oscillator Potential

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ABSTRACT

In this work, we solved the Klein-Gordon equation with quantum mechanical gravitational plus harmonic oscillator potential via the parametric Nikiforov-Uvarov method. The energy equation and the corresponding un-normalized wave function in terms of Laguerre polynomials were obtained. The numerical values for the S–wave was obtained.

Keywords: Klein-Gordon equation; Quantum mechanical gravitational potential; Harmonic oscillator potential; Nikiforov-Uvarov method

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I. Introduction

The bound state solutions of the Klein–Gordon equation (KGE) are only possible for some potentials of physical interest [1]-[5]. These solutions could be exact or approximate and they normally contain all the necessary information for the quantum system. Quite recently, several authors have tried to solve the problem of obtaining exact or approximate solutions of the KG equation for a number of special potentials using different methods [6]–[28]. Some of these potentials are known to play very important roles in many fields of Physics such as Molecular Physics, Solid State and Chemical Physics [29]. When a particle is in a strong potential field, the relativistic effects must be considered, leading to the relativistic quantum mechanical description of such a particle [30]–[34]. In the relativistic limit, the particle's motions are very often described using either the KGE or the Dirac equation depending on the spin character of the particle [30], [32]. The spin-zero particles like the mesons are satisfactorily described by the KGE while the spin-half particles such as the electrons are described by the Dirac equation. It is therefore of interest in nuclear and high energy physics to obtain exact solutions of the KGE and Dirac equation.

The purpose of the present work is to present the solutions of the KGE with the quantum mechanical gravitational potential (QMGP) plus harmonic oscillator potential (HOP) of the form [33] and obtain the S-wave.

$$V(r) = mgr + \delta e^{-kr} + \frac{1}{2}\mu\omega^2 r^2 \tag{1}$$

where r is the displacement, k is momentum, m is the mass, g is gravitational acceleration, δ is an adjustable parameter, μ is the reduced mass and ω is the angular frequency. The QMGP could be used to calculate the energy of a body falling under gravity from quantum mechanical point of view. Berberan-Santos et al [34] have studied the motion of a particle in a gravitational field using the QMGP without the exponential term. They obtained the classical and quantum mechanical position probability distribution function for the particle. The HOP has been widely studied in the literature. For example, Amore and Fernandez [35] studied the two-particle harmonic oscillator in a one – dimensional box and obtained energy eigen values which were comparable with energies from variational and perturbation methods. Jasim et al [36] also investigated the single particle level density in a harmonic – oscillator potential well and obtained very interesting results. They

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studied the properties of the partial level density and the total level density numerically for the harmonic oscillator potential well. Also, Ikot et al [37] derived the energy eigenvalues and the corresponding eigen functions for the two – dimensional harmonic oscillator potential in cartesian and polar coordinates using the Nikiforov –Uvarov method. Xue-Hong et al [38] determined the virial theorem for a class of quantum nonlinear harmonic oscillators.

II. The Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method is based on the solutions of a generalized second-order linear differential equation with special orthogonal functions [39]. The Schrodinger equation of the type as

$$\psi''(r) + [E - V(r)]\psi(r) = 0 \tag{2}$$

can be solved by this method. This can be done by transforming (2) into an equation of hypergeometric type with appropriate coordinate transformation $s = s(r)$ to get:

$$\psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{3}$$

To find the exact solution to Eq. (3), we write $\psi(s)$ as"

$$\psi(s) = \phi(s)\chi(s) \tag{4}$$

Substitution of (4) into (3) yields (5) of hypergeometric type as:

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0 \tag{5}$$

In (4), the wave function $\phi(s)$ is defined as the logarithmic derivative [41]:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{6}$$

with $\pi(s)$ being at most first order polynomials. Also, the hypergeometric-type functions in (5) for a fixed integer n is given by the Rodrigue relation as:

$$\chi_n(s) = \frac{B_n}{\rho_n}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)] \tag{7}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition:

$$\frac{d}{ds}[\sigma^n(s)\rho(s)] = \tau(s)\rho(s) \tag{8}$$

With

$$\tau(s) = \bar{\tau}(s) + 2\pi(s) \tag{9}$$

In order to accomplish the condition imposed on the weight function $\rho(s)$ it is necessary that the polynomial $\tau(s)$ be equal to zero at some point of an interval (a, b) and its derivative at this interval at $\tau'(s) < 0$ will be negative [42]. That is:

$$\frac{d\tau(s)}{ds} < 0 \tag{10}$$

The function $\pi(s)$ and the parameter λ required for the NU method are then defined as [43]:

$$\pi(s) = \frac{\sigma' - \bar{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \bar{\tau}}{2}\right)^2 - \bar{\sigma} + k\sigma} \tag{11}$$

$$\lambda = k + \pi'(s) \tag{12}$$

The s-values in (11) are possible to evaluate if the expression under the square root be square of polynomials. This is possible if and only if its discriminant is zero. Therefore, the new eigenvalues equation becomes:

$$\lambda = \lambda_n = -n\frac{d\tau}{ds} - \frac{n(n-1)}{2}\frac{d^2\sigma}{ds^2}, n = 0,1,2, \dots \tag{13}$$

A comparison between (12) and (13) yields the energy eigenvalues.

Secondly, the parametric generalization of the NU method is expressed by the generalized hypergeometric-type equation:

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1 - c_3 s)^2}[-\xi_1 s^2 + \xi_2 s - \xi_3]\psi(s) = 0 \tag{14}$$

Equation (14) is solved by comparing it with (3) and the following polynomials are obtained:

$$\bar{\tau} = (c_1 - c_2 s), \sigma(s) = s(1 - c_3 s), \bar{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3 \tag{15}$$

Now, substituting (15) into (11) gives:

$$\pi(s) = c_4 + c_5 s \pm [(c_6 - c_3 k_\pm)s^2 + (c_7 + k_\pm)s + c_8]^{1/2} \tag{16}$$

where

$$c_4 = \frac{1}{2}(1 - c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = {c_5}^2 + \xi_1, c_7 = 2c_4c_5 - \xi_2, c_8 = {c_4}^2 + \xi_3 \tag{17}$$

The resulting value of k in (16) is obtained from the condition that the function under the square-root should be square of a polynomial and we get:

$$k_\pm = -(c_7 + 2c_3c_8) \pm 2\sqrt{c_8c_9} \tag{18}$$

where

$$c_9 = c_3c_7 + {c_2}^2c_8 + c_6 \tag{19}$$

The new $\pi(s)$ for k becomes:

$$\pi(s) = c_4 + c_5 s \mp \left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right] \tag{20}$$

k_-value becomes

$$k_- = -(c_7 + 2c_3c_8) - 2\sqrt{c_8c_9} \quad (21)$$

Using (9), we obtain:

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2\begin{bmatrix}(\sqrt{c_9} + c_3\sqrt{c_8})s \\ -\sqrt{c_8}\end{bmatrix} \quad (22)$$

The physical condition for the bound state solution is $\tau' < 0$ and thus:

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0 \quad (23)$$

With the aid of (12) and (13), we obtain the energy equation as:

$$(c_2 - c_3)n + c_3n^2 - (2n+1)c_5 + (2n+1) \times \times (\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (24)$$

The weight function $\rho(s)$ is obtained from (8) as:

$$\rho(s) = s^{c_{10}-1}(1 - c_3s)^{\frac{c_{11}}{c_3} - c_{10} - 1} \quad (25)$$

And together with (7), we have:

$$\chi_n(s) = P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3s) \quad (26)$$

Where

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \quad (27)$$

$P_n^{(\alpha,\beta)}$ are the Jacobi polynomials. The second part of the wave function is obtained from (6) as:

$$\phi(s) = s^{c_{12}}(1 - c_3s)^{-c_{12} - \frac{c_{13}}{c_3}} \quad (28)$$

where

$$c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \quad (29)$$

Thus, the total wave function becomes:

$$\psi(s) = N_n s^{c_{12}}(1 - c_3s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3s) \quad (30)$$

where N_n is the normalization constant.

III. The Klein–Gordon Equation

The three-dimensional KG equation with mixed vector and scalar potentials can be written as [30]:

$$[\nabla^2 + (V(r) - E)^2 - (S(r) + M)^2]\psi(r,\theta,\phi) = 0 \quad (31)$$

where M is the rest mass, E is the relativistic energy, and $S(r)$ and $V(r)$ are the scalar and vector potentials, respectively. ∇^2 is the Laplacian operator, c is the speed of light, and $\hbar$ is the reduced Planck's constant which have been set to unity. In spherical coordinates, the KG equation with the potential $V(r)$ is given as [30]:

$$\begin{bmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ -2(EV(r) + MS(r)) + V^2(r) - S^2(r) + E^2 - M^2 \end{bmatrix}\psi(r,\theta,\phi) = 0 \quad (32)$$

Using the common ansatz for the wave function:

$$\psi(r,\theta,\phi) = \frac{R(r)}{r}Y_{lm}(\theta,\phi) \quad (33)$$

From (32) we get the following set of equations:

$$\frac{d^2R(r)}{dr^2} + \left[E^2 - M^2 - 2(EV(r) + MS(r)) + V^2(r) - S^2(r) - \frac{\lambda}{r^2}\right]R(r) = 0 \quad (34)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + cot\theta\frac{d\Theta(\theta)}{d\theta}\left(\lambda - \frac{m^2}{sin^2\theta}\right)\Theta(\theta) = 0 \quad (35)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi(\phi) = 0 \quad (36)$$

where $\lambda = l(l+1)$ and m^2 are the separation constants. $Y_{lm}(\theta,\phi) = \Theta_{ml}(\theta)\Phi_m(\phi)$ is the solution of (35) and (36) and their solutions are well known as spherical harmonic functions [30]. Equation (34) is the radial part of the KG equation which we are interested in solving. The potential in equation (1) can be written as:

$$V(r) = \beta r + V_0e^{-\alpha r} + \frac{1}{2}\mu\omega^2r^2 \quad (37)$$

where $\beta = mg, \alpha = k, z = r, \delta = V_0$. We can also write (37) as:

$$V(r) = \beta r + V_0(1 - \alpha r + \alpha^2r^2) + \frac{1}{2}\mu\omega^2r^2 \quad (38)$$

On arranging (38) we get our working potential amenable to parametric NU method as:

$$V(r) = V_0 + (\beta - \alpha V_0)r + \left(\alpha^2V_0 + \frac{1}{2}\mu\omega^2\right)r^2 \quad (39)$$

The potential of (39) can be used to solve various quantum mechanical equations including, the Schrodinger equation (SE), Klein-Gordon equation (KGE) and Dirac equation using the NU method for their exact solutions.

IV. Solutions of the Radial Equation

Writing (34) for S-waves and for a special case $V(r) = S(r)$ as:

$$\frac{d^2R(r)}{dr^2} + [E^2 - M^2 - 2(E + M)V(r)]R(r) = 0 \quad (40)$$

And using the potential of (39) in (40) yields:

$$\frac{d^2R(r)}{dr^2}+\left[E^2-M^2-2(E+M)V_0-2(E+M)(\beta-\alpha V_0)r-2(E+M)(\alpha^2V_0+\tfrac{1}{2}\mu\omega^2)r^2\right]R(r)0 \qquad (41)$$

Comparing (41) with (14) yields the following parameters:

$$c_1=c_2=c_3=0, \xi_1=2(E+M)\left(\alpha^2V_0+\tfrac{1}{2}\mu\omega^2\right), \xi_2=-2(E+M)(\beta-\alpha V_0), \xi_3=-(E^2-M^2)+2(E+M)V_0 \qquad (42)$$

Other coefficients are determined as:

$$c_4=\tfrac{1}{2}, c_5=0, c_6=\xi_1, c_7=-\xi_2, c_8=\tfrac{1}{4}+\xi_3, c_9=\xi_1, c_{10}=1+2\sqrt{\tfrac{1}{4}+\xi_3}, c_{11}=2\sqrt{\xi_1}, c_{12}=\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+\xi_3}, c_{13}=-\sqrt{\xi_1} \qquad (43)$$

From (16):

$$\pi(s)=\tfrac{1}{2}\pm\left[\xi_1s^2+(-\xi_2+k_\pm)s+\tfrac{1}{4}+\xi_3\right]^{1/2} \qquad (44)$$

From (18):

$$k_\pm=\xi_2\pm2\sqrt{\left(\tfrac{1}{4}+\xi_3\right)\xi_1} \qquad (45)$$

From (22):

$$\tau(s)=1-\left(2\sqrt{\xi_1}\right)s-\sqrt{\tfrac{1}{4}+\xi_3} \qquad (46)$$

The negative derivative of (46) then becomes:

$$\tau'(s)=-\left(2\sqrt{\xi_1}\right)<0 \qquad (47)$$

The new $\pi(s)$ for the NU method is chosen as:

$$\pi(s)=\tfrac{1}{2}-\left(\sqrt{\xi_1}\right)s-\sqrt{\tfrac{1}{4}+\xi_3} \qquad (48)$$

For

$$k_-=\xi_2-2\sqrt{\left(\tfrac{1}{4}+\xi_3\right)\xi_1} \qquad (49)$$

Now using (24), (42) and (43) we obtain the relativistic limit energy spectrum of the QMGP + HOP as:

$$M^2-E^2=-\left[X-\left(\tfrac{E+M}{2}\right)\mu\omega^2\right]\Big/Y \qquad (50)$$

where

$$X=\left\{(2n+1)\left[2(E+M)\left(\alpha^2V_0+\tfrac{1}{2}\mu\omega^2\right)+2(E+M)(\beta-\alpha V_0)\right]\right\}^2 \qquad (51)$$

$$Y=9\alpha V_0-4\mu\omega^2-4(E-M)\alpha^2V_0-2(E-M)\mu\omega^2 \qquad (52)$$

A. *Non Relativistic Limit*

In this section, we consider the non-relativistic limit of (50). Considering a transformation of the form:

$$M+E_n\rightarrow\frac{2\mu}{\hbar^2} \text{ and } M-E_n\rightarrow-E_n,$$

where μ is the reduced mass, and substituting it into Eq. (50), we have the non-relativistic energy eigenvalues equation as:

$$E_n=\frac{\dfrac{\hbar^2}{2\mu}\left\{2n+1\left[\begin{array}{l}2\left(\alpha^2v_0+\dfrac{1}{2}\mu\omega^2\right)\\+2(\beta-\alpha v_0)\dfrac{2\mu}{\hbar^2}\end{array}\right]\right\}^2-\dfrac{\mu^2\omega^2}{\hbar^2}}{9\alpha v_0-4\mu\omega^2-\dfrac{6\alpha^2v_0\mu}{\hbar^2}-\dfrac{4\mu^2\omega^2}{\hbar^2}} \qquad (53)$$

The weight function $\rho(s)$ is obtained from (25) and the parameters of (43) as:

$$\rho(s)=s^{2\sqrt{\frac{1}{4}+\xi_3}} \qquad (54)$$

and using (26) we get the wavefunction $\chi_n(s)$ as:

$$\chi_n(s)=L_n^{\epsilon}\left(2\sqrt{\xi_1}s\right) \qquad (55)$$

where $\epsilon=2\sqrt{\frac{1}{4}+\xi_3}$ and $L_n^{\epsilon}(2\sqrt{\xi_1}s)$ is the Laguerre polynomial. From (28) the wave function is:

$$\phi(s)=s^{(1+\epsilon)/2} \qquad (56)$$

The radial wave function is then obtained from (30) as:

$$R_{nl}(s)=N_ns^{\frac{1}{2}+\frac{\epsilon}{2}}L_n^{\epsilon}\left(2\sqrt{\xi_1}s\right) \qquad (57)$$

where N_n is the normalization constant.

V. Results and Discussion

TABLE I: Energy level for the Quantum Mechanical Plus Harmonic Potential

n	Energy in (*eV*) For $\alpha=0.2$	Energy in (*eV*) For $\alpha=0.4$	Energy in (*eV*) For $\alpha=0.6$
0	2.338516481	2.240312424	2.456223471
1	3.415549442	3.520937271	3.631148365
2	4.492582404	4.801562118	4.900453713
3	5.569615365	6.082186966	6.143256483
4	6.646648326	7.362811813	7.456723469

We computed numerically the energy spectrum of the quantum mechanical plus harmonic potential using (53) for various values of n with the following parameters of $v_0=0.5\,fm^{-1}$, $\omega=0.5$, $\mu=\hbar=\beta=1$ for $\alpha=0.2,\ 0.4$ and

0.6, respectively as presented in Table I. It is observed that as quantum number is increasing the energy also increases.

VI. Conclusion

In this work, we have obtained the energy eigenvalues and the corresponding un-normalized wavefunction using the parametric NU method for the KGE with the quantum mechanical gravitational and exponential potential plus the harmonic oscillator potential in terms of Laquerre polynomials. We computed the numerical for S- wave and observed that as quantum number increases the energy increases as well.

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2. E. P. Inyang, E. P. Inyang, and M. B. Latif, "A Correlation: TL response of synthetic fused quartz with 60Co gamma (high dose) source and 90Sr/90Y Beta (low dose) source", *Bulletin of Pure and applied Sciences-Physics*, vol. 38(1), pp. 6-12, 2019.

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