Arbitrary Solution of the Schrödinger Equation Interacting with the Superposition of Hulthen with Spin-orbit Plus Adjusted Coulomb Potential Model

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Abstract — In this study, we solve the non-relativistic radial part of the Schrödinger wave equation for the superposition of the Hulthen with spin-orbit plus adjusted Coulomb potential (SHSC) potential using the Nikiforov-Uvarov (NU) method for arbitrary states. The Hulthen with spin-orbit plus adjusted Coulomb (SHSC) potential is the simplest potential field for a nuclear system and has been used to obtain the single particle energy spectrum for both nucleon species orbiting a closed nuclear core. We also obtained in this study the corresponding single particle normalized wave function expressed in terms of the Jacobi polynomial. Besides, we obtained two special cases of the energy spectra for the SHSC potential.

Keywords — Schrodinger equation, spin-orbit interaction, Hulthen potential, Coulomb potential, energy eigenvalue, eigenfunction.

I. INTRODUCTION

There has been a concerted effort to obtain the exact bound – state solution of the wave equation, within relativistic and non-relativistic quantum mechanics. A plethora of problems involving the multidimensional Schrodinger equation have been solved over the past decades. Various analytical methods have been deployed to find a solution to the non-relativistic Schrodinger wave equation. Common amongst these methods are the asymptotic iteration method [1], supersymmetric shape invariance approach [2], exact or proper quantization rule [3], factorization method [4], 1/N shifted expansion method [5] and Nikiforov-Uvarov method [6]. In recent times, the focus has been on combining two or more potentials to model a more compact effective potential that can be applicable to real physical systems. For instance, Agboola [7] obtained the eigenvalues and the corresponding eigen function for Hulthen potential plus the Mie Ring – Shape potential, Alberg and Wilets [8] combined Coulomb and Harmonic Oscillator potentials to find the exact solution of the Schrodinger equation. Worthy of note is the work of Niknam et al [9], which provided the solution to the Schrodinger wave equation by combing Woods-Saxon with spin-orbit interaction and Coulomb potentials as an effective potential.

In the present study, we constructed an effective potential which we named SHSC potential, which models to an acceptable extent the nuclear potential. SHSC is an acronym for the superposition of Hulthen with Spin-orbit interaction and adjusted Coulomb potential.

The Hulthen potential is one of the reputable short range potentials which can easily approximate the Woods-Saxon potential [10] and exhibits the characteristics of the generalized Coulomb potential for small values of r and decreases exponentially for large values of r . The Hulthen potential has been used extensively in many areas of physics including nuclear, atomic, particle and solid state physics. In this work, we present Hulthen potential as a mean field nuclear potential. Here it is used to describe the nucleon-nucleon interaction as well as the interaction between a single nucleon with the nuclear core. However, extensive studies has been carried with Hulthen potential within the relativistic and non-relativistic quantum mechanics respectively [11]-[15].

A more realistic potential for the single particle nuclear system can be obtained by combining the mean – field potential (Hulthen as in our case) [16], spin – orbit interaction [17] and the Coulomb potential (considering the interactions between protons) [18]. This combination along with the centrifugal term forms the effective potential for the nuclear system. The SHSC potential is applied to find the solution to Schrodinger wave equation to obtain the single particle energy spectrum and the corresponding wave function for the nuclear system.

Submitted on November 02, 2021.

Published on December 23, 2021.

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II. REVIEW OF THE NU METHOD

We take a brief overview of the NU method in this section; details can be obtained from ref. [18]. This method involves solving a hyper-geometric type second order differential equation:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0 \quad (1)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are second order polynomials. $\tilde{\tau}(z)$ is a order degree polynomial [19]. A possible solution of Eq. (1) is proposed as:

$$\psi(z) = \phi(z)y(z) \quad (2)$$

which results in a hyper geometric – type equation of the form:

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0 \quad (3)$$

Part of the solution of Eq.(1) given in Eq. (2) as $\phi(z)$ is the solution to another differential equation of the form given by:

$$\sigma(z)\phi'(z) - \pi(z)\phi(z) = 0 \quad (4)$$

where

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) \quad (5)$$

and the parameter λ in Eq. (3) is defined as:

$$\lambda = \lambda_n = -n\tau'(z) + \frac{n(n-1)}{2}\sigma''(z) = 0 (n = 0, 1, 2, 3, \dots) \quad (6)$$

The term $\tau(z)$ is a polynomial, and to obtain a proper solution for the hyper-geometric type differential equation, its first derivative $\tau'(z)$ must be negative. The function $y(z)$ as stated in Eq. (2) is the hyper geometric – type wave function obtained using the Rodrigues relation:

$$y_n(z) = \frac{B_n(z)d^n}{\rho(z)dz^n} [\sigma^n(z)\rho(z)] \quad (7)$$

where B_n is related to the normalization constant, and $\rho(z)$ is defined as:

$$y_n(z) = \frac{B_n(z)d^n}{\rho(z)dz^n} [\sigma^n(z)\rho(z)] \quad (8)$$

Also the function $\pi(z)$ is a first order polynomial is defined as:

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \quad (9)$$

K in Eq. (9) is related to Eq. (6) and the first derivative of $\pi'(z)$ as thus:

$$\lambda = k + \pi'(z) \quad (10)$$

The value of k is obtained by equating the discriminant of the quadratic expression under the square root sign in Eq. (9) to zero. By solving Eq. (6) and Eq. (10), we derive the energy eigenvalue equation.

III. SOLUTION OF SCHRODINGER EQUATION FOR SHSC

The single particle energy spectrum $E_{n\ell j}$ for nuclei is obtained by solving the radial component of the 3-dimensional Schrodinger wave equation given as:

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \frac{2\mu}{\hbar^2} [E_{n\ell j} - V_{\text{eff}}(r)] R(r) = 0 \quad (11)$$

where μ is the reduced mass of a single nucleon, $E_{n\ell j}$ is the energy spectra, $\hbar$ is the Planck's constant, n , ℓ and j are the principal, orbital and total momentum quantum number respectively. Let define a new variable be defined as:

$$z = e^{-2\alpha r} \quad (12)$$

The derivatives of the function $R(r)$ in Eq. (11) are obtained in terms of the new variable z . Equation (11) is transformed and restated in terms of z , as $R(r)$ is redefined as $\psi(z)$. The attempt is to provide an approximate but not exact solution to the Schrodinger equation for the effective potential employed in the present work.

Using the variable as defined by Eq. (12), Eq. (11) is transformed and the Schrodinger equation becomes as below:

$$\begin{aligned} \frac{d^2 \psi(z)}{dz^2} + \frac{1}{z} \frac{d\psi(z)}{dz} + \frac{1}{\alpha^2 z^2} \left[\frac{2\mu E}{\hbar^2} - \frac{2\mu}{\hbar^2} (V_{\text{eff}}) \right] \psi(z) &= 0 \\ \frac{d^2 \psi(z)}{dz^2} + \frac{1}{z} \frac{d\psi(z)}{dz} + \frac{1}{\alpha^2 z^2} \left[\frac{2\mu E}{\hbar^2} - \frac{2\mu}{\hbar^2} (V_{\text{eff}}) \right] \psi(z) &= 0 \end{aligned} \quad (13)$$

The effective potential V_{eff} in Eq. (13) is the SHSC potential which is a superposition of Hulthen, spin-orbit interaction, adjusted Coulomb and the centrifugal term. The Hulthen potential is as defined by Eq. (14).

$$V_H(r) = -\frac{V_0 \alpha e^{-\alpha r}}{1 - e^{-\alpha r}} \quad (14)$$

where V_0 in Eq. (14) is the potential strength of the Hulthen potential.

The adjusted Coulomb potential is given as:

$$V_c(r) = \frac{e^2}{R_0} \left[3 - \left(\frac{r}{R_0} \right)^2 \right] \quad (15)$$

where e is the charge, R_0 the nuclei radius and r is the range of this of the position of a particle from the core.

The spin-orbit interaction is given as:

$$V_{LS}(r) = \frac{1}{2} V_{LS}(0) \left(\frac{r_0}{\hbar} \right)^2 \frac{1}{r} \left[\frac{dV_H(r)}{dr} \right] \vec{L} \cdot \vec{S} \quad (16)$$

$$\text{where } \vec{L} \cdot \vec{S} = \frac{\hbar^2}{2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right)$$

The centrifugal term of the Schrodinger wave equation is given as:

$$V_{\text{cf}} = \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} \quad (17)$$

The SHSC potential V_{eff} for the nuclei system is obtained by combining Eq. (14), Eq. (15), Eq.(16) and Eq.(17) as stated in Eq.(18):

$$V_{\text{eff}} = -\frac{V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} - V_{LS}(0) r_0^2 \frac{1}{r} \frac{\alpha e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \vec{L} \cdot \vec{S} - \frac{3e^2}{\pi \epsilon_0 R_0} + \frac{e^2 r^2}{\pi \epsilon_0 R_0^3} + \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} \quad (18)$$

By the application of the Greene-Aldrich approximations scheme [20], [21], [22] the centrifugal barrier is handled. This approximation scheme is valid for $\alpha \ll 1$. And it is of the following forms:

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \quad (19)$$

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1 - e^{-2\alpha r})^2} \quad (20)$$

Substituting Eq. (18), Eq. (19) and Eq. (20) into Eq. (13), the Schrodinger equation now reads:

$$\frac{d^2\psi(z)}{dz^2} + \frac{1}{z} \frac{d\psi(z)}{dz} + \frac{1}{z^2(1-z)^2} \left[\begin{aligned} &\frac{2\mu E}{\hbar^2 \alpha^2} (1-z)^2 + V_o \frac{2\mu}{\hbar^2 \alpha^2} z(1-z) \\ &+ V_{LS}(0) \frac{2\mu r_o^2}{\hbar^2} \left(\frac{z}{1-z} \right) \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \\ &- \frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} (1-z)^2 + \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} (1-z)^4 \\ &- \ell(\ell+1)z \end{aligned} \right] \psi(z) = 0 \quad (21)$$

By applying the principles of truncation as suggested by [23] to the higher powers of z beyond the second degree, Eq. (21) is transformed to the hyper geometric-type second order differentiate equation solvable by the NU method.

$$\frac{d^2\psi(z)}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{d\psi(z)}{dz} + \frac{1}{z^2(1-z)^2} \left[\begin{aligned} &-\frac{2\mu E}{\hbar^2 \alpha^2} z^2 + V_{LS}(0) \frac{2\mu r_o^2}{\hbar^2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) z^2 \\ &- \frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} z^2 + 6 \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} z^2 - V_o \frac{2\mu}{\hbar^2 \alpha^2} z^2 \\ &+ 2 \frac{2\mu E}{\hbar^2 \alpha^2} z + V_{LS}(0) \frac{2\mu r_o^2}{\hbar^2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) z \\ &+ 2 \frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} z - 4 \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} z - \ell(\ell+1)z \\ &+ V_o \frac{2\mu}{\hbar^2 \alpha^2} z - \frac{2\mu E}{\hbar^2 \alpha^2} - \frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} + \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} \end{aligned} \right] \psi(z) = 0 \quad (22)$$

where

$$\Gamma = \left\{ \begin{aligned} &-\frac{2\mu E}{\hbar^2 \alpha^2} + V_{LS}(0) \frac{2\mu r_o^2}{\hbar^2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \\ &-\frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} + 6 \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} - V_o \frac{2\mu}{\hbar^2 \alpha^2} \end{aligned} \right\} \quad (23)$$

$$\Sigma = \left\{ \begin{aligned} &2 \frac{2\mu E}{\hbar^2 \alpha^2} + V_{LS}(0) \frac{2\mu r_o^2}{\hbar^2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \\ &+ 2 \frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} - 4 \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} + V_o \frac{2\mu}{\hbar^2 \alpha^2} - \ell(\ell+1) \end{aligned} \right\} \quad (24)$$

$$-\mathbb{Q} = \frac{2\mu E}{\hbar^2 \alpha^2} + \frac{6\mu e^2}{R_o \pi \epsilon_0 \alpha^2 \hbar^2} - \frac{2\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_o^3} \quad (25)$$

Putting Eq. (23), Eq. (24) and Eq. (25) into Eq. (22), we have a second differential equation of hypergeometric type Eq. (26):

$$\frac{d^2\psi(z)}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{d\psi(z)}{dz} + \frac{1}{z^2(1-z)^2} [\Gamma z^2 + \Sigma z - \mathbb{Q}] \psi(z) = 0 \quad (26)$$

Comparing Eq. (26) and Eq. (1), the relevant polynomials can be expressed as thus:

$$\begin{aligned}\tilde{\tau}(z) &= 1 - z, & \sigma(z) &= z(1 - z), & \sigma^2(z) &= z^2(1 - z)^2 \\ \sigma(z) &= \Gamma z^2 + \Sigma z - \mathbb{Q}\end{aligned}\quad (27)$$

Inserting the polynomials as defined by Eq. (27) into Eq. (9), we have:

$$\pi(z) = \frac{-z}{2} \pm \sqrt{(\gamma - K)z^2 + (\Sigma + K)z + \mathbb{Q}} \quad (28)$$

where

$$\gamma = \frac{1}{4} + \Gamma \quad (29)$$

As required by the NU method the discriminant of the quadratic expression under the square root sign in Eq. (28) is equated to zero, this result in another quadratic equation in terms of K. solving the resultant quadratic equation for K, we have possible solutions as:

$$K = -(\Sigma + 2\mathbb{Q}) \pm 2\sqrt{\mathbb{Q}}\sqrt{\gamma + \Sigma + \mathbb{Q}} \quad (30)$$

From Eq. (30), substituting the negative solution of K into Eq.(28), we have:

$$\pi(z) = \frac{-z}{2} \pm \left(\sqrt{\mathbb{Q}} + \sqrt{\gamma + \Sigma + \mathbb{Q}}\right)z - \sqrt{\mathbb{Q}} \quad (31)$$

Upon inserting Eq. (31) into Eq. (5) another required polynomial is derived:

$$\tau(z) = 1 - 2z - 2\left(\sqrt{\mathbb{Q}} + \sqrt{\gamma + \Sigma + \mathbb{Q}}\right)z - 2\sqrt{\mathbb{Q}} \quad (32)$$

Differentiating $\tau(z)$ as expressed in Eq. (32) with respect to z we have:

$$\tau'(z) = -2 - 2\sqrt{\mathbb{Q}} - 2\sqrt{\gamma + \Sigma + \mathbb{Q}} \quad (33)$$

Taking the first derivative of Eq. (31) and using Eq. (30), Eq. (33) as well as Eq. (27), λ and λ_n are obtained explicitly as:

$$\lambda = \frac{-1}{2} - \sqrt{\mathbb{Q}} - \sqrt{\gamma + \Sigma + \mathbb{Q}} - \Sigma - 2\mathbb{Q} - 2\sqrt{\mathbb{Q}}\sqrt{\gamma + \Sigma + \mathbb{Q}} \quad (34)$$

$$\lambda_n = n^2 + n + 2n\sqrt{\mathbb{Q}} + 2n\sqrt{\gamma + \Sigma + \mathbb{Q}} \quad (35)$$

By equating Eq. (34) to Eq. (35), we solve for the term $\mathbb{Q}$:

$$\mathbb{Q} = \left[\frac{-\frac{1}{2} \left[\left(n + \frac{1}{2} + \sqrt{\chi} \right)^2 + \frac{1}{4} + \alpha \right]}{\left(n + \frac{1}{2} + \sqrt{\chi} \right)} \right]^2 \quad (36)$$

where

$$\begin{aligned}\chi &= \gamma + \Sigma + \mathbb{Q} \\ \alpha &= -\frac{1}{4} - \gamma + \mathbb{Q}\end{aligned}\quad (37)$$

Reintroducing the Eq. (23), Eq. (24) and Eq. (25) into Eq. (37) and Eq. (36), we obtain the single particle energy spectrum for the SHSC potential in terms of the principal n, orbital ℓ and angular momentum j quantum numbers:

$$E_{n\ell j} = \frac{3e^2}{\pi\epsilon_0 R_o} - \frac{e^2}{\pi\epsilon_0 \alpha^4 R_o^3} - \frac{\hbar^2 \alpha^2}{8\mu} \left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})} - \frac{6\mu e^2}{\hbar^2 \pi\epsilon_0 \alpha^4 R_o^3} + \ell(\ell+1) \right)^2}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})} - \frac{6\mu e^2}{\hbar^2 \pi\epsilon_0 \alpha^4 R_o^3} + \ell(\ell+1) \right)} \right. \\ \left. + \frac{-\frac{2\mu V_o}{\hbar^2 \alpha} + \frac{2\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4}) + \frac{10\mu e^2}{\hbar^2 \pi\epsilon_0 \alpha^4 R_o^3}}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})} - \frac{6\mu e^2}{\hbar^2 \pi\epsilon_0 \alpha^4 R_o^3} + \ell(\ell+1) \right)} \right] \quad (38)$$

Equation (38) is the energy eigenvalue equation for the proton orbiting the core in a nuclear system. The energy spectrum for a neutron orbit the core in a nuclear system is obtained by setting e in Eq. (38) to zero as the net charge of the neutron is zero, hence there is no Coulomb interaction between the orbiting neutron and the core. Hence the energy eigenvalues for the neutron is given as:

$$E_{n\ell j} = -\frac{\hbar^2 \alpha^2}{8\mu} \left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})} + \ell(\ell+1) \right)^2}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})} + \ell(\ell+1) \right)} \right. \\ \left. + \frac{-\frac{2\mu V_o}{\hbar^2 \alpha} + \frac{2\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu V_{LS}(0)r_o^2}{\hbar^2} (j(j+1) - \ell(\ell+1) - \frac{3}{4})} + \ell(\ell+1) \right)} \right] \quad (39)$$

Next we obtain the corresponding single particle eigenfunction as stated in Eq. (8), using the weighted factor $\rho(z)$ which in the form:

$$\rho(z) = z^{2\sqrt{Q}} (1-z)^{2\sqrt{\gamma+\Sigma+Q}} \quad (40)$$

Substituting Eq. (40) into Eq. (7), the Rodrigues relation is obtained as:

$$y_n(z) = B_n z^{-2\sqrt{Q}} (1-z)^{-2\sqrt{\gamma+\Sigma+Q}} \frac{d^n}{dz^n} \left[z^{n+2\sqrt{Q}} (1-z)^{n+2\sqrt{\gamma+\Sigma+Q}} \right] \\ \equiv P_n^{(2\sqrt{Q}, 2\sqrt{\gamma+\Sigma+Q})}(1-2z) \quad (41)$$

where P_n is the Jacobi polynomial. Using Eq. (36) and Eq. (41) the wave function is given as:

$$\psi(z) = N_{n\ell} z^{\sqrt{Q}} (1-z)^{(\frac{1}{2} + \sqrt{\gamma+\Sigma+Q})} P_n^{(2\sqrt{Q}, 2\sqrt{\gamma+\Sigma+Q})}(1-2z) \quad (42)$$

where $N_{n\ell}$ is the normalization constant. Applying the normalization condition the Normalization constant is obtained by applying the normalization condition as thus:

$$\frac{N_{n\ell}^2}{\alpha} \int_{-1}^1 \left(\frac{1-y}{2} \right)^\omega \left(\frac{1+y}{2} \right)^b \left[P_n^{(\omega, b-1)} y \right]^2 dy = 1 \quad (43)$$

Equation (43) is derived by setting:

$$\begin{aligned} b &= \frac{1}{2} + \sqrt{\gamma + \Sigma + \mathbb{Q}} \\ \omega &= 2\sqrt{\mathbb{Q}} \end{aligned} \quad (44)$$

By comparing Eq. (43) to the standard integral of the form Eq. (37) in the ref. [24], we have:

$$\int_{-1}^1 \left(\frac{1-p}{2} \right)^x \left(\frac{1+p}{2} \right)^y \left[P_n^{(2x, 2y-1)} y \right]^2 dp = \frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)} \quad (45)$$

We derive the normalization constant as:

$$N_{n\ell} = \left[\frac{\alpha n! \omega \Gamma(\omega + b + n + 1)}{2\Gamma(\omega + n + 1)\Gamma(b + n + 1)} \right]^{1/2} \quad (46)$$

IV. SPECIAL CASES

A. Case 1: Coulomb potential

The energy spectrum for adjusted Coulomb potential only can be obtained from the energy spectrum for SHSC potential by setting some parameters in Eq. (38) to zero. We set the parameters of SHSC to zero; $V_0 = 0$, $V_{ls} = 0$.

$$E_{n\ell} = \frac{3e^2}{\pi\epsilon_0 R_0} - \frac{e^2}{\pi\epsilon_0 \alpha^4 R_0^3} - \frac{\hbar^2 \alpha^2}{8\mu} \left[\frac{\left[\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{6\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_0^3} + \ell(\ell+1)} \right)^2 + \frac{10\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_0^3} \right]}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{6\mu e^2}{\hbar^2 \pi \epsilon_0 \alpha^4 R_0^3} + \ell(\ell+1)} \right)} \right]^2 \quad (47)$$

Equation (47) is the energy spectrum for the adjusted Coulomb potential.

B. Case 2: Hulthen potential

The energy spectrum for Hulthen potential is obtained from the CHSC potential by setting $V_{ls} = 0$ and $e = 0$;

$$E_{n\ell} = -\frac{\hbar^2 \alpha^2}{8\mu} \left[\frac{(n+1+\ell)^2 - \frac{2\mu V_0}{\hbar^2 \alpha}}{(n+1+\ell)} \right]^2 \quad (48)$$

The special cases of our solution to the Schrodinger equation for SHSC potential has proven that the SHSC potential offers a broad base of application. From our solution equation Eq. (38) and Eq. (39) are applicable to nuclear systems, Eq. (47) gives the Coulomb contribution to the nuclear bound state energy of a proton, while (48) can be applicable to solid state and molecular systems.

V. CONCLUSION

In this work, we have studied in the solution of Schrodinger wave equation for SHSC potential using the Nikiforov-Uvarov method. Applying necessary approximations the single particle energy spectrum for a nucleus with one nucleon orbiting core with a closed shell. We also considered two special cases of the potential and obtain their respective energy spectra. Although the superposition of Hulthen, spin-orbit and adjusted Coulomb potential which we called SHSC potential is novel with respect to its application to the non-relativistic wave equation, the special cases shows that our results is in agreement with literature. As such, our results for the SHSC potential unarguable is valid within reasonable level of approximation. The corresponding normalized single particle wave function for SHSC has been obtain in this study.

ACKNOWLEDGMENT

E. A. Thompson will like to thank E.S. Williams for his encouragement for the successful completion of this work.

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